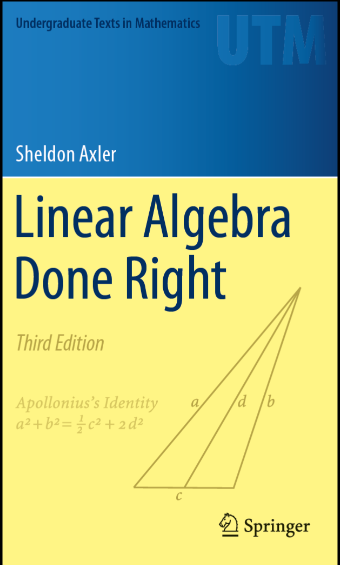
Eigenspaces and Diagonal Matrices



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If an operator has a diagonal matrix with respect to some basis, then the entries along the diagonal are precisely the eigenvalues of the operator.

Eigenspaces

Definition: *eigenspace*, $E(\lambda, T)$

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. The *eigenspace* of T corresponding to λ, denoted $E(\lambda, T)$, is defined by

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Then

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If λ is an eigenvalue of an operator $T \in \mathcal{L}(V)$, then T restricted to $E(\lambda, T)$ is the operator of multiplication by λ.

Sum of Eigenspaces

Sum of eigenspaces is a direct sum

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Suppose also that $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum. Furthermore,

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However, T is diagonalizable, because the matrix of T with respect to the basis $(1, 4), (7, 5)$ is

$$\begin{pmatrix} 69 & 0 \\ 0 & 46 \end{pmatrix},$$

as you should verify.

Conditions equivalent to diagonalizability

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- $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

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Thus T is not diagonalizable.

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With respect to this basis consisting of eigenvectors, T has a diagonal matrix. ∎

Fibonacci [Leonardo of Pisa] (1170 – 1250)

Linear Algebra Done Right, by Sheldon Axler