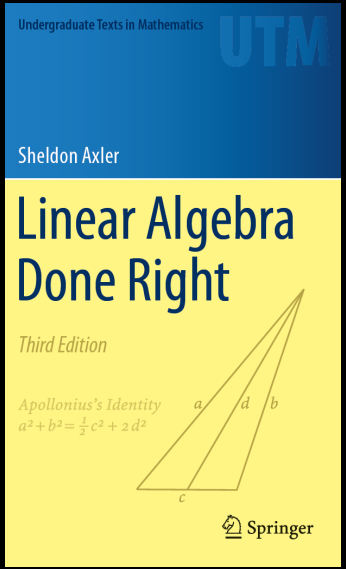
Inner Products and Norms, part 2: Norms



Norms

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Taking square roots now gives the desired equality. ∎

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Woman Teaching Geometry

Cauchy–Schwarz Inequality

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Examples:

- If $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbf{R}$, then

$$|x_1 y_1 + \cdots + x_n y_n|^2 \leq (x_1{}^2 + \cdots + x_n{}^2)(y_1{}^2 + \cdots + y_n{}^2).$$

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- If f, g are continuous real-valued functions on $[-1, 1]$, then
$$\left| \int_{-1}^{1} f(x) g(x) \, dx \right|^2 \leq \left(\int_{-1}^{1} \big(f(x)\big)^2 \, dx \right) \left(\int_{-1}^{1} \big(g(x)\big)^2 \, dx \right).$$

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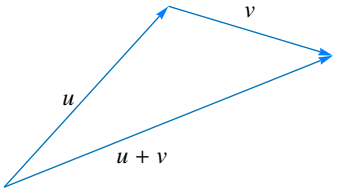

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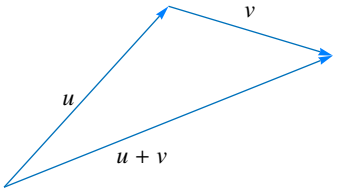

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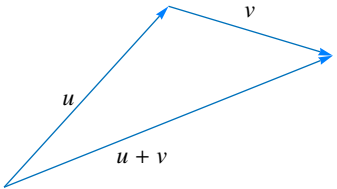

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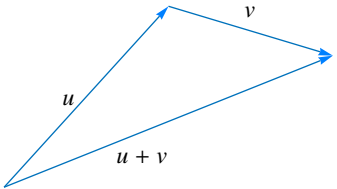

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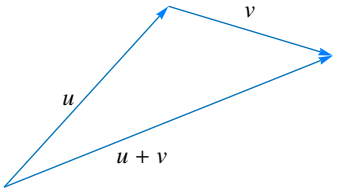

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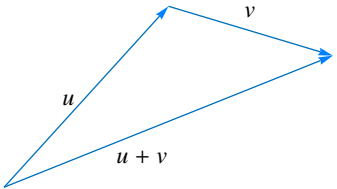

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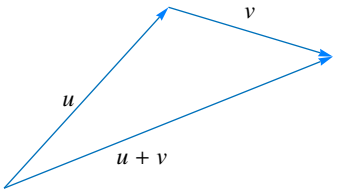

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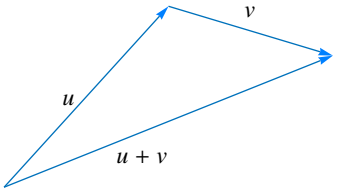

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&= (\|u\| + \|v\|)^2
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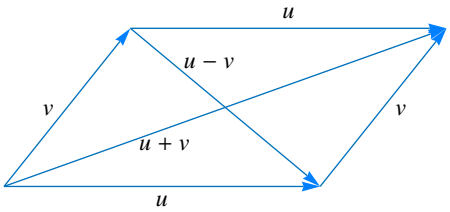

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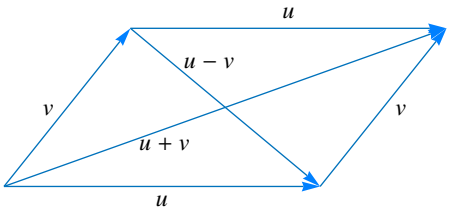

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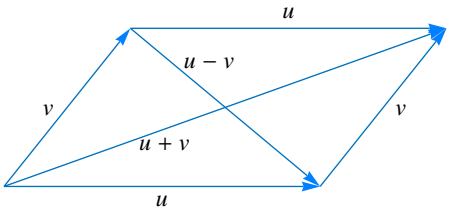

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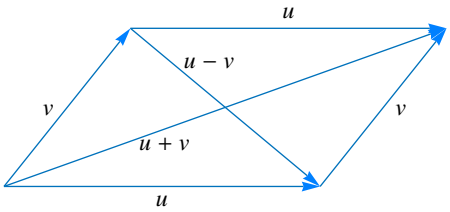

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&\qquad = \|u\|^2 + \|v\|^2 + \langle u, v\rangle + \langle v, u\rangle \\
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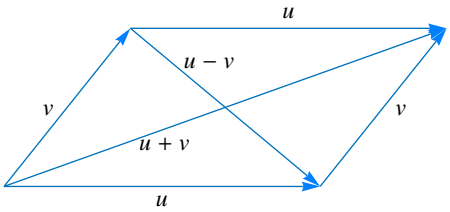

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&\quad = \|u\|^2 + \|v\|^2 + \langle u, v\rangle + \langle v, u\rangle \\
&\qquad + \|u\|^2 + \|v\|^2 - \langle u, v\rangle - \langle v, u\rangle \\
&\quad = 2(\|u\|^2 + \|v\|^2)
\end{aligned}$$

Linear Algebra Done Right, by Sheldon Axler