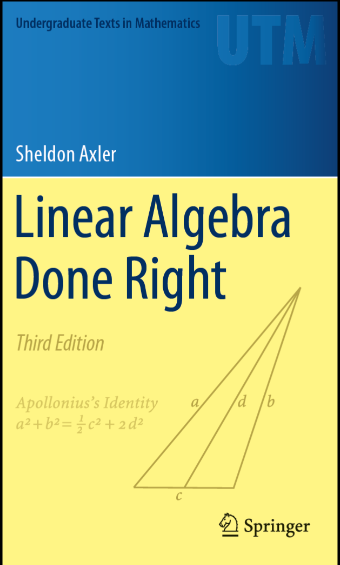
Null Spaces and Ranges



Notation

$\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.

V and W denote vector spaces over $\mathbf{F}$.

Null Space

Definition: *null space,* $\operatorname{null} T$

For $T \in \mathcal{L}(V, W)$, the *null space* of T, denoted $\operatorname{null} T$, is the subset of V consisting of those vectors that T maps to 0:

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- Suppose $\varphi \in \mathcal{L}(\mathbf{C}^3, \mathbf{C})$ is defined by $\varphi(z_1, z_2, z_3) = z_1 + 2z_2 + 3z_3$. Then $\operatorname{null} \varphi = \{(z_1, z_2, z_3) \in \mathbf{C}^3 : z_1 + 2z_2 + 3z_3 = 0\}$.

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- Suppose $D \in \mathcal{L}\big(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R})\big)$ is the differentiation map defined by $Dp = p'$. The only functions whose derivative equals the zero function are the constant functions. Thus the null space of D equals the set of constant functions.

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Injectivity is equivalent to null space equals $\{0\}$

Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if $\operatorname{null} T = \{0\}$.

Range

Definition: *range*

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- Suppose $T \in \mathcal{L}(\mathbf{R}^2, \mathbf{R}^3)$ is defined by $T(x, y) = (2x, 5y, x + y)$, then $\operatorname{range} T = \{(2x, 5y, x + y) : x, y \in \mathbf{R}\}$.

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- Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbf{R}), \mathcal{P}(\mathbf{R}))$ is the differentiation map defined by $Dp = p'$. Because for every polynomial $q \in \mathcal{P}(\mathbf{R})$ there exists a polynomial $p \in \mathcal{P}(\mathbf{R})$ such that $p' = q$, the range of D is $\mathcal{P}(\mathbf{R})$.

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Definition: ***surjective***

A function $T \colon V \to W$ is called *surjective* if its range equals W.

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We do this by proving that $Tv_1, \ldots, Tv_n$ is a basis of $\operatorname{range} T$. ∎

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A map to a smaller dimensional space is not injective

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Homogeneous Equations

Fix positive integers m and n, and let $A_{j,k} \in \mathbf{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$. Consider the homogeneous system of linear equations

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$$T(x_1, \ldots, x_n) = \Big(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k\Big).$$

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A homogeneous system of linear equations with more variables than equations has nonzero solutions.

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Inhomogeneous Equations

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An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.

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Carl Friedrich Gauss (1777 – 1855)

Linear Algebra Done Right, by Sheldon Axler