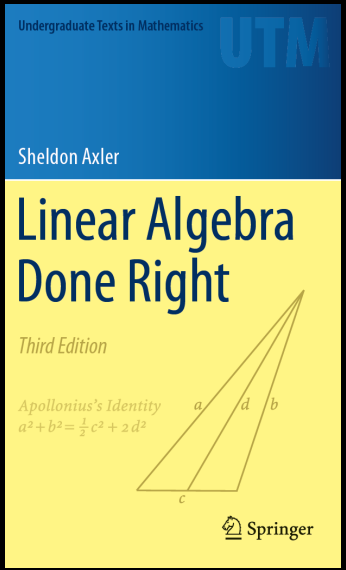
Polynomials



Notation

$\mathbf{F}$ denotes either $\mathbf{R}$ or $\mathbf{C}$.

Real Part, Imaginary Part

Definition: $\operatorname{Re} z$, $\operatorname{Im} z$

Suppose $z = a + bi$, where a and b are real numbers.

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Example: Suppose $z = 3 + 2i$. Then

$$\operatorname{Re} z = 3 \quad \text{and} \quad \operatorname{Im} z = 2.$$

Complex Conjugate, Absolute Value

Definition: *complex conjugate, $\bar{z}$, absolute value, $|z|$*

Suppose $z \in \mathbf{C}$.

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Example: Suppose $z = 3 + 2i$. Then

- $\bar{z} = 3 - 2i$;
- $|z| = \sqrt{3^2 + 2^2} = \sqrt{13}$.

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Properties of complex numbers

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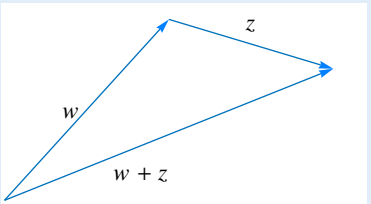
- $\bar{\bar{z}} = z$;
- $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$
- $|\bar{z}| = |z|$;
- $|wz| = |w|\,|z|$;
- Triangle Inequality: $|w + z| \leq |w| + |z|$.



Zero Polynomials

Definition: *polynomial,* $\mathcal{P}(\mathbf{F})$

- A function $p\colon \mathbf{F} \to \mathbf{F}$ is called a *polynomial* with coefficients in $\mathbf{F}$ if there exist $a_0, \ldots, a_m \in \mathbf{F}$ such that

  $$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

  for all $z \in \mathbf{F}$.

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If a polynomial is the zero function, then all its coefficients are 0

Suppose $a_0, \ldots, a_m \in \mathbf{F}$. If

$$a_0 + a_1 z + \cdots + a_m z^m = 0$$

for every $z \in \mathbf{F}$, then $a_0 = \cdots = a_m = 0$.

Division Algorithm for Polynomials

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Suppose that $p, s \in \mathcal{P}(\mathbf{F})$, with $s \neq 0$. Then there exist unique polynomials $q, r \in \mathcal{P}(\mathbf{F})$ such that

$$p = sq + r$$

and $\deg r < \deg s$.

Zeros of Polynomials

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A number $\lambda \in \mathbf{F}$ is called a *zero* (or *root*) of a polynomial $p \in \mathcal{P}(\mathbf{F})$ if

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Each zero of a polynomial corresponds to a degree-1 factor

Suppose $p \in \mathcal{P}(\mathbf{F})$ and $\lambda \in \mathbf{F}$. Then $p(\lambda) = 0$ if and only if there is a polynomial $q \in \mathcal{P}(\mathbf{F})$ such that

$$p(z) = (z - \lambda)q(z)$$

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A polynomial has at most as many zeros as its degree

Suppose $p \in \mathcal{P}(\mathbf{F})$ is a polynomial with degree $m \geq 0$. Then p has at most m distinct zeros in $\mathbf{F}$.

Omar Khayyám (1048 – 1131)

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Every nonconstant polynomial with complex coefficients has a zero.

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Factorization of a polynomial over $\mathbf{C}$

If $p \in \mathcal{P}(\mathbf{C})$ is a nonconstant polynomial, then p has a unique factorization (except for the order of the factors) of the form

$$p(z) = c(z - \lambda_1) \cdots (z - \lambda_m),$$

where $c, \lambda_1, \ldots, \lambda_m \in \mathbf{C}$.

Complex Zeros of Polynomials with Real Coefficients

Polynomials with real coefficients have zeros in pairs

Suppose $p \in \mathcal{P}(\mathbf{C})$ is a polynomial with real coefficients. If $\lambda \in \mathbf{C}$ is a zero of p, then so is $\bar{\lambda}$.

Factorization of Polynomials over $\mathbf{R}$

Factorization of a quadratic polynomial

Suppose $b, c \in \mathbf{R}$. Then there is a polynomial factorization of the form

$$x^2 + bx + c = (x - \lambda_1)(x - \lambda_2)$$

with $\lambda_1, \lambda_2 \in \mathbf{R}$ if and only if $b^2 \geq 4c$.

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Suppose $p \in \mathcal{P}(\mathbf{R})$ is a nonconstant polynomial. Then p has a unique factorization (except for the order of the factors) of the form

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1 x + c_1) \cdots (x^2 + b_M x + c_M),$$

where $c, \lambda_1, \ldots, \lambda_m, b_1, \ldots, b_M, c_1, \ldots, c_M \in \mathbf{R}$, with ${b_j}^2 < 4c_j$ for each j.

Linear Algebra Done Right, by Sheldon Axler