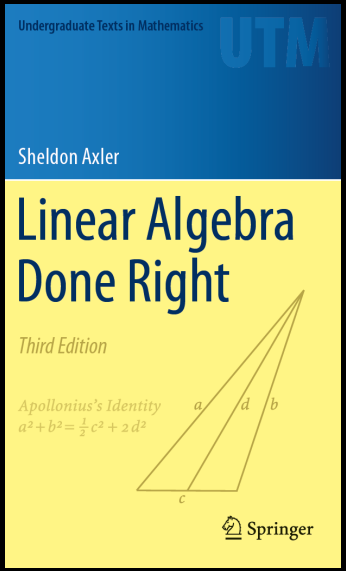
$\mathbf{R}^n$ and $\mathbf{C}^n$



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Thus we invent a solution, called i, with the property that $i^2 = -1$.

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If $a \in \mathbf{R}$, we identify $a + 0i$ with the real number a. Thus we think of $\mathbf{R}$ as a subset of $\mathbf{C}$. We also usually write $0 + bi$ as just bi, and we usually write $0 + 1i$ as just i. The definition of multiplication shows that $i^2 = -1$.

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Elements of $\mathbf{F}$ are sometimes called *scalars*, which is just a fancy word for numbers.

Definition: $\mathbf{R}^2$ *and* $\mathbf{R}^3$

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- The set $\mathbf{R}^3$, which you can think of as ordinary space, is the set of all ordered triples of real numbers:

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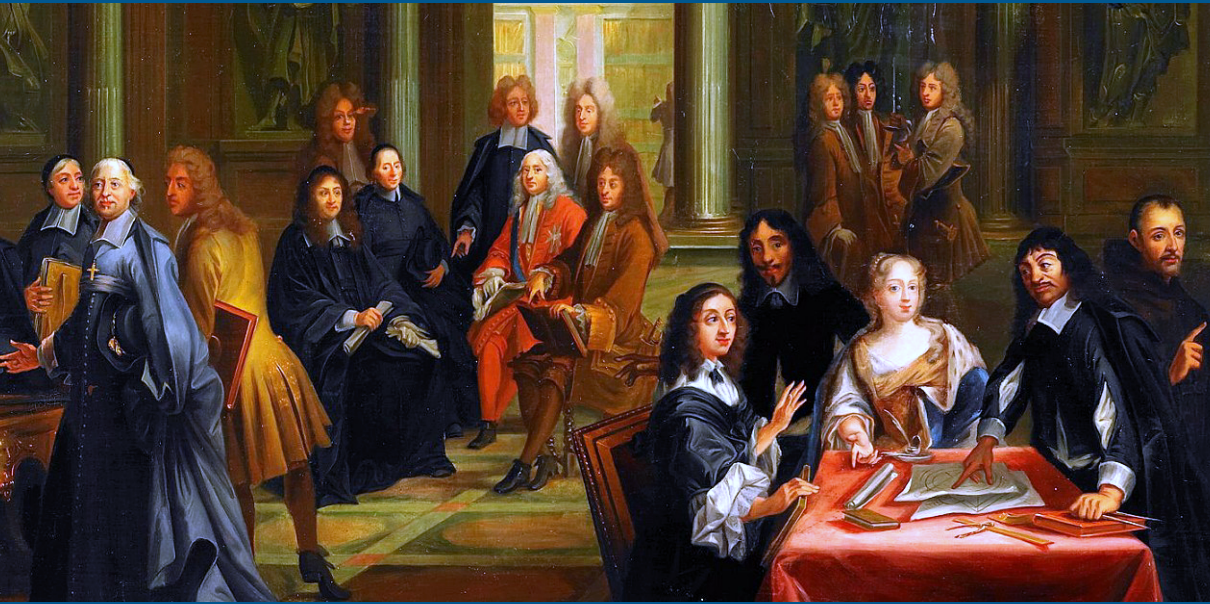
René Descartes and Queen Christina



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Two lists are equal if and only if they have the same length and the same elements in the same order.

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Elements of $\mathbf{F}^n$ are often called *points* or *vectors*.

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Addition in $\mathbf{F}^n$ is defined by adding corresponding coordinates:

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Example: $i(i, 2, 5 + i, -3i) = (-1, 2i, -1 + 5i, 3)$

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Linear Algebra Done Right, by Sheldon Axler